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Geometrically Nonlinear Theory of Multilayered Plates with Interlayer Slips

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The formulation of a geometrically nonlinear theory of anisotropic multilayered plates of general layups featuring interlayer slips is discussed. The theory rests on a displacement field, which accounts for an arbitrary distribution of the tangential displacements through the laminate thickness, fulfills a priori the static continuity conditions of tangential stresses at the layer interfaces, and allows for jumps in the tangential displacements so as to provide the possibility of incorporating effects of interfacial imperfection. For the interlayer displacement jump, a linear shear slip law is postulated. No a priori assumption is made on the type and order of the expansion in the thicknesswise direction of the tangential displacements. The pertinent equations of motion and consistent boundary conditions are derived by means of the dynamic version of the principle of virtual work. These are given in terms of force and moment stress resultants and in terms of generalized displacements. The generalization achieved by the proposed approach is shown by deriving, as particular cases, the recently proposed first-order and third-order models for laminated plates featuring interlayer slips.

I. Introduction

IT is well known that, unlike their homogeneous isotropic counterparts, the anisotropic constitution of multilayered composite structures often results in unique phenomena that can occur at vastly different geometric scales, i.e., at the global level, at the ply level, or at the reinforcement-matrix level. Because the properties of composite materials are significantly influenced by the properties of interfaces between the constituents, the perfect interface assumption (featuring continuous displacements and tractions across the interface and, thus, disregarding the interface properties and structures) used in most analytical and numerical work on composite materials in many cases of interest could be inadequate. For example, one of the possible ways of yielding stiffness degradation in laminated composite structures is typically associated with the interlayer slip. The interlayer slip causes separation of bonded layers and, as a result, stiffness degradation. This, in fact, constitutes a fracture mode for the laminated structures with negative repercussions on the overall behavior of the structure, which involve its static, dynamic, and stability responses. Whereas the technical literature of the last two decades registers a growing number of papers on modeling of laminated solid and sandwich beams, plates, and shells with perfectly bonded layers (for example, the survey paper of Kapania and Raciti,[1,2] Refs. 3–7, and the recent book by Reddy[8]); surprisingly, in spite of its importance, the theory of laminated structures with nonrigidly bonded interfaces was developed for the most part for laminated beams,[9–11] with very few research investigations on the topic of the present paper, e.g., Refs. 12–15. The aim of the present paper is to make a contribution in this field.

The paper is concerned with the formulation of a geometrically nonlinear theory (in the von Kármán sense) for laminated composite plates of general lamination configurations and featuring nonrigidly bonded interfaces. The approach makes use of a displacement field, which 1) accounts for an arbitrary distribution of the tangential displacements through the laminate thickness, 2) fulfills a priori the static continuity conditions of tangential stresses at the layer interfaces, and 3) allows for jumps in the tangential displacements so as to provide the possibility of incorporating effects of interfacial imperfection. For the interlayer displacement jump a linear shear slip law is postulated, which includes also the two extreme cases of perfectly bonded surfaces and of completely debonded ones. [Note that it has been shown by Lu and Liu[13] that the linear slip theory can be made equivalent to a very thin embedded compliant layer approach. Also, the embedded layer approach appears to work well in modeling delamination or imperfect bond.[11] The author is indebted to one of the referees for calling attention to these two papers.] The pertinent equations of motion and consistent boundary conditions are derived by means of the dynamic version of the principle of virtual work in terms of force and moment stress resultants. After deriving the constitutive equations for arbitrarily laminated plates, the equations of motion also are given in terms of generalized displacements. Other key features of the proposed approach are as follows: 1) no a priori assumption is made on the type and order of the expansion in the thicknesswise direction of the tangential displacements; 2) the number of generalized displacement in the kinematics is independent of the number of layers, as in the equivalent single-layer theories; and 3) if present, the assumed kinematics takes into account the unsymmetry in the layup; that is, in the assumed displacement field there are some parameters whose value depends on the layup and are zero for symmetric layup. The paper represents a continuation and development of the results obtained previously by Di Sciuva.[16]

The generalization achieved by the proposed approach is shown by deriving, as particular cases, the recently proposed first-order[14] and third-order[15] models for laminated plate featuring interlayer slips. It is hoped that the presented developments will contribute to a deeper understanding and reliable prediction of the load carrying capacity and failure of such structures.

II. Preliminaries

Consider a composite laminated plate consisting of a finite number of linearly elastic anisotropic parallel layers, each of them exhibiting different physicomechanical properties. The existence of the imperfect bonding between the surfaces of two arbitrary contiguous layers is postulated.

$\mathcal{V}$ is the volume of the plate in the undeformed (reference) configuration, Ω^+ and Ω^- are the upper and bottom external planes of the plate, whereas $\mathcal{S}$ is the lateral boundary surface of $\mathcal{V}$ generated by the normal to Ω along its boundary curve Γ (with arc length s). Moreover, $\mathcal{S}_p$ and $\mathcal{S}_u$ and Γ_p and Γ_u are the two parts of $\mathcal{S}$ and Γ, where tractions and displacements, respectively, are prescribed. The thickness of the kth lamina and of the entire plate are denoted as ${}^{(k)}h$ $(k = 1, 2, \ldots, N)$ and h, respectively, where N is the total number of layers.

The thickness of each layer, as well as of the entire plate, is assumed to be constant, and the material of each layer is assumed to

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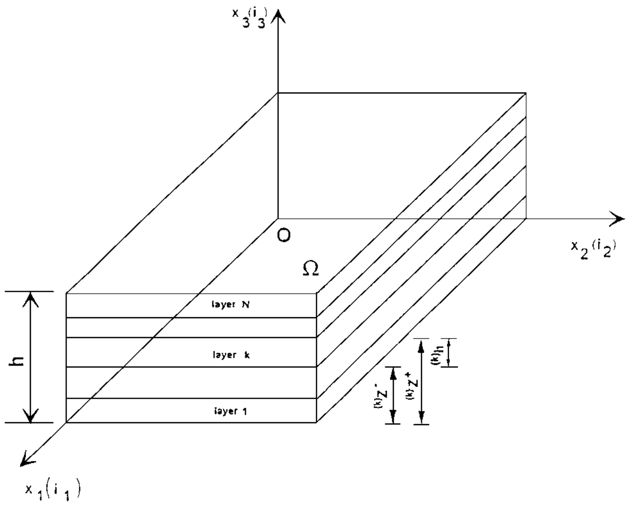

Fig. 1 Plate geometry and coordinate system.

possess a plane of elastic symmetry parallel to the reference surface. For the sake of convenience, the undeformed bottom surface of the plate is selected as the reference plane Ω (Fig. 1). The points of the three-dimensional plate structure are referred to an orthogonal Cartesian coordinate system x_j $(j = 1, 2, 3)$ with unit vectors $\boldsymbol{i}_j$, where x_α $(\alpha = 1, 2)$ is the set of in-plane coordinates on Ω and x_3 is the coordinate normal to Ω.

The distances (along x_3) between the reference plane and the undeformed upper and bottom faces of the kth layer are denoted by $^{(k)}Z^+$ and $^{(k)}Z^-$, respectively. Associated with the boundary curve Γ of Ω, we define the unit tangent and outward unit normal vectors $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively, by

$$\boldsymbol{t} = t_\alpha \boldsymbol{i}_\alpha, \qquad \boldsymbol{n} = n_\alpha \boldsymbol{i}_\alpha = \boldsymbol{t} \times \boldsymbol{i}_3 \tag{1}$$

where t_α and n_α are the direction cosines of unit tangent and the unit outward normal with respect to the α axis and $\boldsymbol{n}$, $\boldsymbol{t}$, and $\boldsymbol{i}_3$ are oriented in such a way that the (n, t, x_3) form a right-handed coordinate system. Unless otherwise specified, the usual Cartesian indicial notation is employed with Latin indices ranging from 1 to 3 and Greek indices ranging from 1 to 2, respectively. Repeated indices imply the Einsteinian summation convention, and $(\cdot\cdot)_{,i}$ is used to denote partial differentiation with respect to x_i. Superscript (k) placed to the right of any quantity identifies its affiliation to the kth layer, whereas superscript (k) placed to the left of any quantity identifies the element in the series expansion of the displacement components. Moreover, $\tilde{\sigma}_{ij}$ is the second Piola–Kirchhoff stress tensor and $\tilde{e}_{ij}$ the Lagrange strain tensor. Under the assumptions that each layer possesses a plane of elastic symmetry parallel to the reference plane (x_1, x_2) and that $\sigma_{33} = 0$, the following stress-strain relations hold for each layer:

$$\tilde{\sigma}_{\alpha\beta} = Q_{\alpha\beta\gamma\delta}\tilde{e}_{\gamma\delta} + \tilde{\lambda}_{\alpha\beta}\Theta, \qquad \tilde{\sigma}_{\alpha 3} = 2Q_{\alpha 3\gamma 3}\tilde{e}_{\gamma 3} \tag{2}$$

where $Q_{\alpha\beta\gamma\delta}$ are the transformed reduced components of the stiffness tensor; they are symmetric in the label α and β, γ and δ, and the pairs of indices $\alpha\beta$ and $\gamma\delta$. Here $\tilde{\lambda}_{\alpha\beta}$ are the transformed thermal expansion coefficients, and $\Theta = \Theta(x_i)$ denotes the stationary temperature rise from a reference value. It is postulated that the elastic properties are temperature independent.

III. Kinematics

In the spirit of the von Kármán partially nonlinear theory, the following expressions for the Lagrangian strain-displacement relationships are used:

$$\tilde{e}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i} + \tilde{u}_{3,i}\tilde{u}_{3,j}) \tag{3}$$

where $\tilde{u}_i(x_j)$ is the displacement in the x_i direction.

Let us consider a laminated composite plate featuring interlamina slips. We know from the theory of elasticity that the displacements and stresses at the interface k between the kth and $(k+1)$th perfectly bonded layers must satisfy the following geometric and static continuity conditions (contact conditions):

$$^{(k)}\tilde{u}_\alpha\big|_{x_3 = {}^{(k)}Z^+} = {}^{(k+1)}\tilde{u}_\alpha\big|_{x_3 = {}^{(k+1)}Z^-} \tag{4}$$

$$\tilde{\sigma}_{\alpha 3}\big|_{x_3 = {}^{(k)}Z^+} = \tilde{\sigma}_{\alpha 3}\big|_{x_3 = {}^{(k+1)}Z^-} \tag{5}$$

whereas at the interface l between the lth and $(l+1)$th layers featuring interlamina slip, Eq. (4) should be substituted by the following, expressing the interlayer jump of tangential displacement components:

$$^{(l)}\hat{U}_\alpha = {}^{(l+1)}\tilde{u}_\alpha\big|_{x_3 = {}^{(l+1)}Z^-} - {}^{(l)}\tilde{u}_\alpha\big|_{x_3 = {}^{(l)}Z^+} \tag{6}$$

If the thickness distributions of the transverse shear and normal strains are assumed to be arbitrary, then the contact conditions on the stresses will, in general, not be satisfied if the layers have different mechanical properties. In developing a displacement field that fulfills the contact conditions (5) on the transverse shearing stresses, we assume the following expressions for $\tilde{u}_i(x_j)$ in the kth layer:

$$\begin{aligned} \tilde{u}_\alpha(x_j) &= u_\alpha(x_j) + U_\alpha(x_j) + \hat{U}_\alpha(x_j) \\ \tilde{u}_3(x_j) &= u_3^{(0)}(x_\beta) \end{aligned} \tag{7}$$

where

$$u_\alpha(x_j) = \sum_{r=0}^{R} L^{(r)}(x_3)u_\alpha^{(r)}(x_\beta) \tag{8}$$

gives the contribution to the in-plane displacement, which is continuous with respect to the thickness coordinate x_3 (this is the classical series expansion used in the single-layer or smeared laminate models),

$$U_\alpha(x_j) = \sum_{k=1}^{N-1} {}^{(k)}\phi_\alpha(x_\beta)\big(x_3 - {}^{(k)}Z^+\big)H_k \tag{9}$$

gives the contribution to the in-plane displacement, which is continuous with respect to x_3 but with jumps in the first derivative at the interfaces between adjacent layers (this is the expansion used to model multilayered plates by enforcing the continuity of the transverse shear stresses at the interfaces in the zig-zag models,[17–20] and

$$\hat{U}_\alpha(x_j) = \sum_{k=1}^{N-1} {}^{(k)}\hat{U}_\alpha(x_\beta)H_k \tag{10}$$

gives the in-plane displacement jumps across each interface, which enable interfacial imperfection to be incorporated. Here, $H_k = H(x_3 - {}^{(k)}Z^+)$ is the Heaviside unit function and $^{(k)}\phi_\alpha(x_\beta)$ are functions to be determined by satisfying the contact conditions (5) on the transverse shearing stresses at the interfaces.

The functions $L^{(r)}(x_3)$ can be given by any set of linearly independent functions, at least continuous with their first derivatives with respect to x_3 (see Sec. V). Substitution of the displacement field (7) into Eq. (3) yields

$$\begin{aligned} 2\tilde{e}_{\alpha\beta} = {} & \sum_{r=0}^{R} L^{(r)}\big(u_{\beta,\alpha}^{(r)} + u_{\alpha,\beta}^{(r)}\big) + u_{3,\alpha}^{(0)}u_{3,\beta}^{(0)} + \sum_{k=1}^{N-1}\big({}^{(k)}\phi_{\beta,\alpha} + {}^{(k)}\phi_{\alpha,\beta}\big) \\ & \times \big(x_3 - {}^{(k)}Z^+\big)H_k + \sum_{k=1}^{N-1}\big({}^{(k)}\hat{U}_{\beta,\alpha} + {}^{(k)}\hat{U}_{\alpha,\beta}\big)H_k \end{aligned} \tag{11}$$

$$2\tilde{e}_{\alpha 3} = \sum_{r=0}^{R} u_\alpha^{(r)}L_{,3}^{(r)} + u_{3,\alpha}^{(0)} + \sum_{k=1}^{N-1} {}^{(k)}\phi_\alpha H_k$$

We compute $^{(k)}\phi_\alpha$ in such a way that continuity conditions on the transverse shearing stresses are satisfied at the interface. The result is

$$^{(k)}\phi_\alpha = {}^{(k)}a_{\alpha\beta}u_{3,\beta}^{(0)} + \sum_{r=0}^{R} {}^{(k)}a_{\alpha\beta}^{(r)}u_\beta^{(r)} \tag{12}$$

where ${}^{(k)}a_{\alpha\beta}$, ${}^{(k)}a^{(r)}_{\alpha\beta}$ are known constants depending only on the transverse shear mechanical properties of the constituent layers,

$$ {}^{(k)}a_{\alpha\beta} = {}^{(k)}A_{\alpha\beta} + {}^{(k)}A_{\alpha\delta} \sum_{q=1}^{k-1} {}^{(q)}a_{\delta\beta} \tag{13} $$

$$ {}^{(k)}a^{(r)}_{\alpha\beta} = {}^{(k)}A_{\alpha\beta} L^{(r)}_{,3}\big|_{x_3={}^{(k)}Z^+} + {}^{(k)}A_{\alpha\delta} \sum_{q=1}^{k-1} {}^{(q)}a^{(r)}_{\delta\beta} \tag{14} $$

with

$$ {}^{(k)}A_{\alpha\beta} = {}^{(k+1)}S_{\alpha 3\gamma 3}\big({}^{(k+1)}Q_{\gamma 3\beta 3} - {}^{(k)}Q_{\gamma 3\beta 3}\big) \tag{15} $$

$Q_{\alpha 3\beta 3}$ and $S_{\alpha 3\beta 3}$ are the components of the transverse shear stiffness and compliance tensors, respectively. Then, we obtain

$$ U_\alpha(x_j) = \sum_{k=1}^{N-1}\left[{}^{(k)}a_{\alpha\beta}u^{(0)}_{3,\beta} + \sum_{r=0}^{R} {}^{(k)}a^{(r)}_{\alpha\beta}u^{(r)}_\beta\right]\big(x_3 - {}^{(k)}Z^+\big)H_k \tag{16} $$

and, consequently,

$$ \tilde{u}_\alpha(x_j) = \sum_{r=0}^{R} L^{(r)}(x_3)u^{(r)}_\alpha(x_\tau) + \sum_{k=1}^{N-1}\left[{}^{(k)}a_{\alpha\beta}u^{(0)}_{3,\tau} + \sum_{r=0}^{R} {}^{(k)}a^{(r)}_{\alpha\beta}u^{(r)}_\beta\right] \times\big(x_3 - {}^{(k)}Z^+\big)H_k + \sum_{k=1}^{N-1} {}^{(k)}\hat{U}_\alpha(x_\tau)H_k \tag{17} $$

or, in equivalent form,

$$ \tilde{u}_\alpha(x_j) = \sum_{r=0}^{R}\left[\delta_{\alpha\beta}L^{(r)}(x_3) + \sum_{k=1}^{N-1} {}^{(k)}a^{(r)}_{\alpha\beta}\big(x_3 - {}^{(k)}Z^+\big)H_k\right]u^{(r)}_\beta(x_\tau) + u^{(0)}_{3,\beta}\sum_{k=1}^{N-1} {}^{(k)}a_{\alpha\beta}\big(x_3 - {}^{(k)}Z^+\big)H_k + \sum_{k=1}^{N-1} {}^{(k)}\hat{U}_\alpha(x_\tau)H_k \tag{18} $$

Let

$$ P^{(r)}_{\alpha\beta}(x_3) = \delta_{\alpha\beta}L^{(r)}(x_3) + \sum_{k=1}^{N-1} {}^{(k)}a^{(r)}_{\alpha\beta}\big(x_3 - {}^{(k)}Z^+\big)H_k \tag{19} $$

$$ h_{\alpha\beta}(x_3) = \sum_{k=1}^{N-1} {}^{(k)}a_{\alpha\beta}\big(x_3 - {}^{(k)}Z^+\big)H_k \tag{20} $$

then Eq. (18) is

$$ \tilde{u}_\alpha(x_j) = \sum_{r=0}^{R} P^{(r)}_{\alpha\beta}(x_3)u^{(r)}_\beta(x_\tau) + h_{\alpha\beta}(x_3)u^{(0)}_{3,\beta} + \sum_{k=1}^{N-1} {}^{(k)}\hat{U}_\alpha(x_\beta)H_k \tag{21} $$

which is a generalization to unsymmetric multilayered plates featuring interlayer slips of our previous results on unsymmetric multilayered plates with perfectly bonded layers.[16]

For the interlayer displacement jump, we will postulate a linear shear slip law (see Ref. 15 for a broad discussion on this topic)

$$ {}^{(k)}\hat{U}_\alpha\big(x_\tau, x_3 = {}^{(k)}Z^+\big) = {}^{(k)}T_{\alpha\beta}\big(x_\tau, x_3 = {}^{(k)}Z^+\big) \times {}^{(k)}\tilde{\sigma}_{\beta 3}\big(x_\tau, x_3 = {}^{(k)}Z^+\big) \tag{22} $$

where ${}^{(k)}T_{\alpha\beta} \geq 0$ are the sliding constants (spring–layer interface) between the kth and $(k+1)$th layers. In addition to the extreme situations corresponding to the rigidly bonding interfaces (${}^{(k)}T_{\alpha\beta} = 0$ yielding ${}^{(k)}\hat{U}_\alpha = 0$) and completely debonding interfaces (${}^{(k)}T_{\alpha\beta} = \infty$ yielding ${}^{(k)}\tilde{\sigma}_{\alpha 3} = 0$), Eq. (22) also covers the case of imperfectly bonded interfaces (${}^{(k)}T_{\alpha\beta} \neq 0, \infty$). Obviously, Eq. (22) is appropriate to model only the sliding deformation because for a completely debonded interfaces the general deformation also includes opening of debonded faces.

With the use of the stress–strain relations (2) and by taking into account that

$$ 2\tilde{e}_{\alpha 3} = \sum_{r=0}^{R} P^{(r)}_{\alpha\beta,3}u^{(r)}_\beta + (\delta_{\alpha\beta} + h_{\alpha\beta,3})u^{(0)}_{3,\beta} $$

Eq. (22) is

$$ {}^{(k)}\hat{U}_\alpha = {}^{(k)}T_{\alpha\lambda}(x_\omega)\,{}^{(k)}Q_{\lambda 3\gamma 3} \times\left[\sum_{r=0}^{R} P^{(r)}_{\gamma\beta,3}u^{(r)}_\beta + (\delta_{\gamma\beta} + h_{\gamma\beta,3})u^{(0)}_{3,\beta}\right]_{x_3={}^{(k)}Z^+} \tag{23} $$

[Note that, in general, ${}^{(k)}T_{\alpha\lambda}(x_\tau)$ will be a function of the in-plane coordinate x_τ. We will consider ${}^{(k)}T_{\alpha\lambda}$ constant with respect to x_τ.] Then, Eq. (21) is

$$ \tilde{u}_\alpha(x_j) = \sum_{r=0}^{R} \hat{P}^{(r)}_{\alpha\beta}(x_3)u^{(r)}_\beta(x_\tau) + \hat{h}_{\alpha\beta}(x_3)u^{(0)}_{3,\beta}(x_\tau) \tag{24} $$

where we have posed

$$ \hat{P}^{(r)}_{\alpha\beta}(x_3) = P^{(r)}_{\alpha\beta}(x_3) + \sum_{k=1}^{N-1} {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda 3\gamma 3}P^{(r)}_{\gamma\beta,3}\big({}^{(k)}Z^+\big)H_k \tag{25} $$

$$ \hat{h}_{\alpha\beta}(x_3) = h_{\alpha\beta}(x_3) + \sum_{k=1}^{N-1} {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda 3\gamma 3}\big[\delta_{\gamma\beta} + h_{\gamma\beta,3}\big({}^{(k)}Z^+\big)\big]H_k \tag{26} $$

For the strain components, we obtain

$$ 2\tilde{e}_{\alpha\beta} = \sum_{r=0}^{R}\big[\hat{P}^{(r)}_{\alpha\gamma}u^{(r)}_{\gamma,\beta} + \hat{P}^{(r)}_{\beta\gamma}u^{(r)}_{\gamma,\alpha}\big] + \hat{h}_{\alpha\gamma}u^{(0)}_{3,\gamma\beta} + \hat{h}_{\beta\gamma}u^{(0)}_{3,\gamma\alpha} + u^{(0)}_{3,\alpha}u^{(0)}_{3,\beta} \tag{27} $$

$$ 2\tilde{e}_{\alpha 3} = \sum_{r=0}^{R} P^{(r)}_{\alpha\beta,3}u^{(r)}_\beta + (\delta_{\alpha\beta} + h_{\alpha\beta,3})u^{(0)}_{3,\beta} \tag{28} $$

Notice that Eq. (24) is simply a series expansion in x_3 of $\tilde{u}_\alpha$ with basis functions having discontinuous first derivative with respect to x_3 at the interfaces between layers but continuous in the interior of the layers.

IV. Equations of Motion and Boundary Conditions

The equations of motion and the variationally consistent boundary conditions are formulated in a weak form using the dynamic version of the principle of virtual displacements,

$$ \int_\Omega \langle\tilde{\sigma}_{\alpha\beta}\delta\tilde{e}_{\alpha\beta} + 2\tilde{\sigma}_{\alpha 3}\delta\tilde{e}_{\alpha 3}\rangle\,d\Omega = \int_\Omega \big(\bar{p}^t_3 + \bar{p}^b_3\big)\delta u^{(0)}_3\,d\Omega + \sum_{l=1}^{N}\int_{{}^{(l)}\mathcal{S}_p} \bar{p}_i\delta\tilde{u}_i\,d\mathcal{S} - \int_\Omega \langle\rho\ddot{\tilde{u}}_i\delta\tilde{u}_i\rangle\,d\Omega \tag{29} $$

where

$$ \langle\cdots\rangle = \sum_{k=1}^{N}\int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+}(\cdots)\,dx_3 $$

Here, ${}^{(k)}\mathcal{S}_p$ is the portion of the lateral cylindrical surface of the kth layer on which the external loads ${}^{(k)}\bar{p}_i$ are assigned; $\bar{p}^t_3$ and $\bar{p}^b_3$ are the transverse loads applied on the top Ω^+ and bottom Ω^- surfaces of the plate; ρ is the material mass density; the overdot indicates differentiation with respect to the time, and the overbar the prescribed value of a quantity.

Making use in Eq. (29) of the strain-displacement relations (27) and (28), and applying Green's theorem wherever feasible, one obtain the following expression for the virtual work statement (29):

$$ -\sum_{s=0}^{R}\int_\Omega\left(\hat{S}^{(s)}_{\alpha\gamma,\alpha} - T^{(s)}_\gamma - \sum_{r=0}^{R}\hat{m}^{(r,s)}_{\beta\gamma}\ddot{u}^{(r)}_\beta - \hat{m}^{(s)}_{\gamma\beta}\ddot{u}^{(0)}_{3,\beta}\right)\delta u^{(s)}_\gamma\,d\Omega + \int_\Omega\left[\hat{M}_{\alpha\gamma,\gamma\alpha} - (Q_\alpha + T_\alpha)_{,\alpha} - \big(N_{\alpha\beta}u^{(0)}_{3,\alpha}\big)_{,\beta} - \bar{p}^t_3 - \bar{p}^b_3 - \sum_{r=0}^{R}\hat{m}^{(r)}_{\beta\gamma}\ddot{u}^{(r)}_{\beta,\gamma} - \hat{m}_{\beta\gamma}\ddot{u}^{(0)}_{3,\beta\gamma} + m^{(0)}\ddot{u}^{(0)}_3\right]\delta u^{(0)}_3\,d\Omega $$

$$-\int_{\Gamma_p}\left[\sum_{s=0}^{R}\bar{\hat{S}}_{\gamma n}^{(s)}\delta u_\gamma^{(s)}+(\bar{V}_3-\bar{\hat{M}}_{nt,n})\delta u_3^{(0)}+\bar{\hat{M}}_{nn}\delta u_{3,n}^{(0)}\right]\mathrm{d}\Gamma$$

$$-\left[\bar{\hat{M}}_{nt}\delta u_3^{(0)}\right]_{|\Gamma_p}+\int_\Gamma\left[\sum_{r=0}^{R}\hat{S}_{\gamma n}^{(r)}\delta u_\gamma^{(r)}\right.$$

$$\left.+\left(N_{\alpha n}u_{3,\alpha}^{(0)}-\hat{V}_3-\hat{M}_{nt,t}\right)\delta u_3^{(0)}+\hat{M}_{nn}\delta u_{3,n}^{(0)}\right]\mathrm{d}\Gamma+\left[\hat{M}_{nt}\delta u_3^{(0)}\right]_{|\Gamma}$$

$$+\int_\Gamma n_\gamma\left(Q_\gamma+T_\gamma+\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r)}\ddot{u}_\beta^{(r)}+\hat{m}_{\beta\gamma}\ddot{u}_{3,\beta}^{(0)}\right)\delta u_3^{(0)}\,\mathrm{d}\Gamma=0 \tag{30}$$

where

$$\hat{V}_3=\hat{M}_{\alpha\gamma,\gamma}n_\alpha \tag{31}$$

and

$$(\cdot)_{\alpha n}=n_\beta(\cdot)_{\alpha\beta},\qquad (\cdot)_n=n_\alpha(\cdot)_{\alpha n},\qquad (\cdot)_{nt}=n_1(\cdot)_{2n}-n_2(\cdot)_{1n} \tag{32}$$

on Γ. In Eq. (30),

$$\left(N_{\alpha\beta};\,\hat{M}_{\alpha\beta};\,\hat{S}_{\alpha\beta}^{(r)}\right)=\left\langle\tilde{\sigma}_{\alpha\gamma}\left(\delta_{\gamma\beta};\,\hat{h}_{\gamma\beta};\,\hat{P}_{\gamma\beta}^{(r)}\right)\right\rangle \tag{33}$$

$$\left(T_\beta;\,Q_\beta;\,T_\beta^{(r)}\right)=\left\langle\tilde{\sigma}_{\alpha 3}\left(\delta_{\alpha\beta};\,h_{\alpha\beta,3};\,P_{\alpha\beta,3}^{(r)}\right)\right\rangle \tag{34}$$

are force and moment stress resultants for unit length,

$$\left(m^{(i)};\,\hat{m}_{\beta\gamma};\,\hat{m}_{\beta\gamma}^{(r)};\,\hat{m}_{\beta\gamma}^{(r,s)}\right)=\left\langle\rho\left(x_3^i;\,\hat{h}_{\alpha\beta}\hat{h}_{\alpha\gamma};\,\hat{P}_{\alpha\beta}^{(r)}\hat{h}_{\alpha\gamma};\,\hat{P}_{\alpha\beta}^{(r)}\hat{P}_{\alpha\gamma}^{(s)}\right)\right\rangle \tag{35}$$

are inertia resultants, and

$$\left(\bar{\hat{M}}_{\gamma n};\,\bar{\hat{S}}_{\gamma n}^{(s)}\right)=\left\langle\bar{p}_\alpha\left(\hat{h}_{\alpha\gamma};\,\hat{P}_{\alpha\gamma}^{(s)}\right)\right\rangle;\qquad \bar{V}_3=\langle\bar{p}_3\rangle \tag{36}$$

are resultants of the applied tractions.

A. Equations of Motion

Setting the coefficients of the virtual variations of the generalized coordinates in the domain integrals equal to zero yields the following equations of motion.

For $\delta u_\gamma^{(s)}$,

$$\hat{S}_{\alpha\gamma,\alpha}^{(s)}-T_\gamma^{(s)}=\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r,s)}\ddot{u}_\beta^{(r)}+\hat{m}_{\gamma\beta}^{(s)}\ddot{u}_{3,\beta}^{(0)},\qquad s=1,\ldots,R \tag{37}$$

For $\delta u_3^{(0)}$,

$$\hat{M}_{\alpha\gamma,\gamma\alpha}-(Q_\alpha+T_\alpha)_{,\alpha}-\left(N_{\alpha\beta}u_{3,\alpha}^{(0)}\right)_{,\beta}=\bar{p}_3^t+\bar{p}_3^b$$

$$+\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r)}\ddot{u}_{\beta,\gamma}^{(r)}+\hat{m}_{\beta\gamma}\ddot{u}_{3,\beta\gamma}^{(0)}-m^{(0)}\ddot{u}_3^{(0)} \tag{38}$$

The preceding equations are given in terms of force and moment stress resultants. To express them in terms of the generalized displacement coordinates, we need the plate constitutive equations. Substituting Eqs. (27) and (28) into the stress–strain relations (2) and the resulting ones into Eqs. (33) and (34) yields the following plate constitutive equations:

$$\left(N_{\alpha\beta};\,\hat{M}_{\alpha\beta};\,\hat{S}_{\alpha\beta}^{(r)}\right)=\sum_{r=0}^{R}\left(\hat{D}_{\alpha\beta\tau\mu}^{(r)};\,\hat{E}_{\alpha\beta\tau\mu}^{(r)};\,\hat{F}_{\alpha\beta\tau\mu}^{(s,r)}\right)u_{\tau,\mu}^{(r)}$$

$$+\left(\hat{B}_{\alpha\beta\tau\mu};\,\hat{C}_{\alpha\beta\tau\mu};\,\hat{E}_{\alpha\beta\tau\mu}^{(s)}\right)u_{3,\tau\mu}^{(0)}+\frac{1}{2}\left(\hat{A}_{\alpha\beta\tau\mu};\,\hat{\mathcal{B}}_{\alpha\beta\tau\mu};\,\hat{D}_{\alpha\beta\tau\mu}^{(s)}\right)$$

$$\times u_{3,\tau}^{(0)}u_{3,\mu}^{(0)}+\left(N_{\alpha\beta}^{\Theta};\,\hat{M}_{\alpha\beta}^{\Theta};\,\hat{S}_{\alpha\beta}^{\Theta^{(r)}}\right) \tag{39}$$

$$\left(T_\alpha;\,Q_\alpha;\,T_\alpha^{(s)}\right)=\sum_{r=0}^{R}\left(D_{\alpha 3\tau 3}^{(r)};\,E_{\alpha 3\tau 3}^{(r)};\,F_{\alpha 3\tau 3}^{(s,r)}\right)u_\tau^{(r)}$$

$$+(\hat{A}_{\alpha 3\tau 3}+\hat{B}_{\alpha 3\tau 3};\,\hat{B}_{\tau 3\alpha 3}+\hat{C}_{\alpha 3\tau 3};\,\hat{D}_{\tau 3\alpha 3}+\hat{E}_{\tau 3\alpha 3})u_{3,\tau}^{(0)} \tag{40}$$

where

$$\begin{Bmatrix}\hat{A}_{\alpha\beta\tau\mu}\\ \hat{B}_{\alpha\beta\tau\mu}\\ \hat{D}_{\alpha\beta\tau\mu}^{(r)}\end{Bmatrix}=\left\langle Q_{\alpha\gamma\pi\mu}\delta_{\gamma\beta}\begin{Bmatrix}\delta_{\pi\tau}\\ \hat{h}_{\pi\tau}\\ \hat{P}_{\pi\tau}^{(r)}\end{Bmatrix}\right\rangle \tag{41}$$

$$\begin{Bmatrix}\hat{\mathcal{B}}_{\alpha\beta\tau\mu}\\ \hat{C}_{\alpha\beta\tau\mu}\\ \hat{E}_{\alpha\beta\tau\mu}^{(r)}\end{Bmatrix}=\left\langle Q_{\alpha\gamma\pi\mu}\hat{h}_{\gamma\beta}\begin{Bmatrix}\delta_{\pi\tau}\\ \hat{h}_{\pi\tau}\\ \hat{P}_{\pi\tau}^{(r)}\end{Bmatrix}\right\rangle \tag{42}$$

$$\hat{F}_{\alpha\beta\tau\mu}^{(r,s)}=\left\langle Q_{\alpha\gamma\pi\mu}\hat{P}_{\gamma\beta}^{(r)}\hat{P}_{\pi\tau}^{(s)}\right\rangle \tag{43}$$

$$\begin{Bmatrix}\hat{A}_{\beta 3\tau 3}\\ \hat{B}_{\beta 3\tau 3}\\ \hat{D}_{\beta 3\tau 3}^{(r)}\end{Bmatrix}=\left\langle Q_{\alpha 3\gamma 3}\delta_{\alpha\beta}\begin{Bmatrix}\delta_{\gamma\tau}\\ \hat{h}_{\gamma\tau,3}\\ \hat{P}_{\gamma\tau,3}^{(r)}\end{Bmatrix}\right\rangle \tag{44}$$

$$\begin{Bmatrix}\hat{C}_{\beta 3\tau 3}\\ \hat{E}_{\beta 3\tau 3}\end{Bmatrix}=\left\langle Q_{\alpha 3\gamma 3}\hat{h}_{\alpha\beta,3}\begin{Bmatrix}\hat{h}_{\gamma\tau,3}\\ \hat{P}_{\gamma\tau,3}^{(r)}\end{Bmatrix}\right\rangle \tag{45}$$

$$F_{\beta 3\tau 3}^{(r,s)}=\left\langle Q_{\alpha 3\gamma 3}P_{\alpha\beta,3}^{(r)}P_{\gamma\tau,3}^{(r)}\right\rangle \tag{46}$$

are the plate stiffnesses and

$$\left(N_{\alpha\beta}^{\Theta};\,\hat{M}_{\alpha\beta}^{\Theta};\,\hat{S}_{\alpha\beta}^{\Theta^{(r)}}\right)=\left\langle\tilde{\lambda}_{\alpha\gamma}\Theta\left(\delta_{\gamma\beta};\,\hat{h}_{\gamma\beta};\,\hat{P}_{\gamma\beta}^{(r)}\right)\right\rangle$$

are the thermal stress resultants and stress couples. In deriving the preceding relations, use has been made of the symmetry conditions of the elasticity coefficients.

By taking into account Eqs. (39) and (40), the equations of motion can be expressed in terms of generalized displacement components as follows.

For $\delta u_\gamma^{(s)}$,

$$\sum_{r=0}^{R}\hat{F}_{\alpha\gamma\tau\mu}^{(s,r)}u_{\tau,\mu\alpha}^{(r)}+\hat{E}_{\alpha\gamma\tau\mu}^{(s)}u_{3,\tau\mu\alpha}^{(0)}+\hat{D}_{\alpha\gamma\pi\mu}^{(s)}\left(u_{3,\pi\alpha}^{(0)}u_{3,\mu}^{(0)}+u_{3,\pi}^{(0)}u_{3,\mu\alpha}^{(0)}\right)$$

$$-\sum_{r=0}^{R}F_{\gamma 3\tau 3}^{(s,r)}u_\tau^{(r)}+(D_{\tau 3\gamma 3}+E_{\tau 3\gamma 3})u_{3,\tau}^{(0)}+\hat{S}_{\alpha\gamma,\alpha}^{\Theta^{(s)}}$$

$$=\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r,s)}\ddot{u}_\beta^{(r)}+\hat{m}_{\gamma\beta}^{(s)}\ddot{u}_{3,\beta}^{(0)} \tag{47}$$

For $\delta u_3^{(0)}$,

$$\sum_{r=0}^{R}\hat{E}_{\alpha\gamma\tau\mu}^{(r)}u_{\tau,\mu\gamma\alpha}^{(r)}+\hat{C}_{\alpha\gamma\tau\pi}u_{3,\tau\pi\gamma\alpha}^{(0)}+\frac{1}{2}\hat{\mathcal{B}}_{\alpha\gamma\pi\mu}$$

$$\times\left(u_{3,\pi\gamma\alpha}^{(0)}u_{3,\mu}^{(0)}+u_{3,\pi}^{(0)}u_{3,\mu\gamma\alpha}^{(0)}\right)-\left(E_{\alpha 3\tau 3}^{(r)}+D_{\alpha 3\tau 3}^{(r)}\right)u_{\tau,\alpha}^{(r)}$$

$$-(\hat{A}_{\alpha 3\tau 3}+\hat{B}_{\tau 3\alpha 3}+\hat{B}_{\alpha 3\tau 3}+\hat{C}_{\alpha 3\tau 3})u_{3,\tau\alpha}^{(0)}-\left(N_{\alpha\beta}u_{3,\alpha}^{(0)}\right)_{,\beta}$$

$$+\hat{M}_{\alpha\gamma,\gamma\alpha}^{\Theta}=\bar{p}_3^t+\bar{p}_3^b-\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r)}\ddot{u}_{\beta,\gamma}^{(r)}-\hat{m}_{\beta\gamma}\ddot{u}_{3,\beta\gamma}^{(0)}+m^{(0)}\ddot{u}_3^{(0)} \tag{48}$$

B. Boundary Conditions

The set of boundary conditions is

$$u_\gamma^{(r)}=\overline{u_\gamma^{(r)}} \tag{49a}$$

$$\hat{S}_{\gamma n}^{(r)}=\overline{\hat{S}_{\gamma n}^{(r)}} \tag{49b}$$

$$u_3^{(0)}=\overline{u_3^{(0)}} \tag{50a}$$

$$N_{\alpha n}u_{3,\alpha}^{(0)}-\hat{V}_3-\hat{M}_{nt,t}+n_\gamma(Q_\gamma+T_\gamma)$$

$$=\bar{V}_3-\overline{\hat{M}}_{nt,t}-n_\gamma\left(\sum_{r=0}^{R}\hat{m}_{\beta\gamma}^{(r)}+\hat{m}_{\beta\gamma}\ddot{u}_{,\beta}^{(0)}\right) \tag{50b}$$

$$u_{3,n}^{(0)}=\overline{u_{3,n}^{(0)}} \tag{51a}$$

$$\hat{M}_{nn}=\overline{\hat{M}}_{nn} \tag{51b}$$

where Eqs. (49a), (50a), and (51a) are prescribed on Γ_u and Eqs. (49b), (50b), and (51b) are natural on Γ_p.

V. Special Plate Models

To show the generalization achieved by the preceding formulation, in this section we obtain, by specialization, two plate models recently proposed in the literature for laminated plate featuring interlayer slips. To do this, we start by noting that, with minor exceptions,[21] in the literature the set of basis funtions $L^{(r)}(x_3)$ is commonly chosen to be polynomial functions, specifically, power expansion of x_3. In accordance with this trend, let us choose

$$L^{(r)}(x_3) = x_3^r, \qquad r = 0, 1, \ldots, R \tag{52}$$

Note that these functions are linearly independent but not orthogonal. It is well known that it is possible to construct a set of polynomials from these that are orthogonal on the interval $[-h/2, +h/2]$. These polynomials are the Legendre polynomials of the first kind of the argument $z = 2x_3/h$,

$$L^{(r)}(z) = \frac{1}{2^r r}\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^2-1)^r$$

As a consequence of Eq. (52),

$$\hat{P}^{(0)}_{\alpha\beta} = \delta_{\alpha\beta} \tag{53}$$

$$\hat{P}^{(1)}_{\alpha\beta} = \delta_{\alpha\beta}x_3 + \sum_{k=1}^{N-1} {}^{(k)}a_{\alpha\beta}\left(x_3 - {}^{(k)}Z^+\right)H_k + \sum_{k=1}^{N-1} {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda3\gamma3}\left(\delta_{\gamma\beta} + \sum_{p=1}^{k-1} {}^{(p)}a_{\gamma\beta}H_p\right)H_k \tag{54}$$

$$\hat{h}_{\alpha\beta} = \sum_{k=1}^{N-1} {}^{(k)}a_{\alpha\beta}\left(x_3 - {}^{(k)}Z^+\right)H_k + \sum_{k=1}^{N-1} {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda3\gamma3}\left(\delta_{\gamma\beta} + \sum_{p=1}^{k-1} {}^{(p)}a_{\gamma\beta}H_p\right)H_k \tag{55}$$

Comparing Eq. (54) with Eq. (55), we obtain

$$\hat{P}^{(1)}_{\alpha\beta} = x_3\delta_{\alpha\beta} + \hat{h}_{\alpha\beta} = x_3\hat{P}^{(0)}_{\alpha\beta} + \hat{h}_{\alpha\beta} \tag{56}$$

Thus, Eq. (24) transforms to

$$\tilde{u}_\alpha(x_j) = u^{(0)}_\alpha + x_3u^{(1)}_\alpha + \left(u^{(1)}_\beta + u^{(0)}_{3,\beta}\right)\hat{h}_{\alpha\beta} + \sum_{r=2}^{R}\hat{P}^{(r)}_{\alpha\beta}u^{(r)}_\beta \tag{57}$$

and

$$\begin{aligned}2\tilde{e}_{\alpha\beta} &= u^{(0)}_{\alpha,\beta} + u^{(0)}_{\beta,\alpha} + x_3\left(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha}\right) + \left(u^{(1)}_\gamma + u^{(0)}_{3,\gamma}\right)_{,\beta}\hat{h}_{\alpha\gamma} + \left(u^{(1)}_\gamma + u^{(0)}_{3,\gamma}\right)_{,\alpha}\hat{h}_{\beta\gamma}\\ &\quad + \sum_{r=2}^{R}\left(\hat{P}^{(r)}_{\alpha\gamma}u^{(r)}_{\gamma,\beta} + \hat{P}^{(r)}_{\beta\gamma}u^{(r)}_{\gamma,\alpha}\right) + u^{(0)}_{3,\alpha}u^{(0)}_{3,\beta}\end{aligned} \tag{58}$$

$$2\tilde{e}_{\alpha3} = \sum_{r=2}^{R}P^{(r)}_{\alpha\beta,3}u^{(r)}_\beta + \left(u^{(1)}_\beta + u^{(0)}_{3,\beta}\right)\left[\delta_{\alpha\beta} + \sum_{k=1}^{N-1}{}^{(k)}a_{\alpha\beta}H_k\right] \tag{59}$$

By taking into account Eqs. (56) and (33), it follows that

$$\hat{S}^{(0)}_{\alpha\beta} = N_{\alpha\beta}, \qquad \hat{S}^{(1)}_{\alpha\beta} = M^{(1)}_{\alpha\beta} + \hat{M}_{\alpha\beta} \tag{60}$$

where we have posed

$$M^{(1)}_{\alpha\beta} = \langle\tilde{\sigma}_{\alpha\gamma}\delta_{\gamma\beta}x_3\rangle \tag{61}$$

Similarly, by taking into account that

$$P^{(0)}_{\alpha\beta,3} = 0, \qquad P^{(1)}_{\alpha\beta,3} = \delta_{\alpha\beta} + h_{\alpha\beta,3}$$

it follows from Eq. (34) that

$$T^{(0)}_\beta = 0, \qquad T^{(1)}_\beta = T_\beta + Q_\beta \tag{62}$$

Now, let us impose the following constraints:

$$\tilde{\epsilon}_{\alpha3} = 0 \qquad \text{for} \qquad x_3 = 0, h \tag{63}$$

(Note that, for the multilayered plates considered in this study, this implies $\tilde{\sigma}_{\alpha3} = 0$ for $x_3 = 0, h$.)

By taking into account that in the first layer the continuity constants are zero, we find that the constraint (63) will be met if

$$u^{(1)}_\beta = -u^{(0)}_{3,\beta} \tag{64a}$$

$$\sum_{r=2}^{R}A^{(r)}_{\alpha\beta}u^{(r)}_\beta = 0 \tag{64b}$$

where we have posed

$$A^{(r)}_{\alpha\beta} = P^{(r)}_{\alpha\beta,3}\big|_{x_3=h} = rh^{r-1}\delta_{\alpha\beta} + \sum_{k=1}^{N-1}{}^{(k)}a^{(r)}_{\alpha\beta}, \qquad r = 2, 3 \tag{65}$$

Equation (64b) allows us to express $u^{(p)}_\alpha$ in terms of $u^{(q)}_\alpha$, $q = 2, \ldots, R$; $q \neq p$. The result is

$$u^{(p)}_\alpha = \mathcal{C}^{(pq)}_{\alpha\beta}u^{(q)}_\beta, \qquad p \neq q \tag{66}$$

with

$$\mathcal{C}^{(pq)}_{\alpha\beta} = -A^{(p)}_{\alpha\gamma^-}A^{(q)}_{\gamma\beta} \tag{67}$$

where $A^{(r)}_{\alpha\beta^-}$ are the elements of the inverse of the square matrix (2×2) whose elements are $A^{(r)}_{\alpha\beta}$. Note that, in the smeared laminate approach, $A^{(r)}_{\alpha\beta} = rh^{r-1}\delta_{\alpha\beta}$. As a result, we obtain

$$\mathcal{C}^{(pq)}_{\alpha\beta} = -\frac{qh^{q-1}}{ph^{p-1}}\delta_{\alpha\beta}$$

If we choose $p = 2$ in Eq. (66), then Eq. (57) after imposing the constraint condition (63) will be

$$\begin{aligned}\tilde{u}_\alpha(x_j) &= u^{(0)}_\alpha - x_3u^{(0)}_{3,\alpha} + \sum_{r=2}^{R}\hat{P}^{(r)}_{\alpha\beta}u^{(r)}_\beta\\ &= u^{(0)}_\alpha - x_3u^{(0)}_{3,\alpha} + \sum_{r=3}^{R}\mathcal{L}^{(r)}_{\alpha\beta}u^{(r)}_\beta\end{aligned} \tag{68}$$

where we have made the following position:

$$\mathcal{L}^{(r)}_{\alpha\beta} = \hat{P}^{(2)}_{\alpha\omega}\mathcal{C}^{(2r)}_{\omega\beta} + \hat{P}^{(r)}_{\alpha\beta} \qquad \text{for} \qquad r \geq 3 \tag{69}$$

For the strain components, we obtain

$$\begin{aligned}2\tilde{e}_{\alpha\beta} &= u^{(0)}_{\alpha,\beta} + u^{(0)}_{\beta,\alpha} - 2x_3u^{(0)}_{3,\alpha\beta} + \sum_{r=2}^{R}\left(\hat{P}^{(r)}_{\alpha\gamma}u^{(r)}_{\gamma,\beta} + \hat{P}^{(r)}_{\beta\gamma}u^{(r)}_{\gamma,\alpha}\right) + u^{(0)}_{3,\alpha}u^{(0)}_{3,\beta}\\ &= u^{(0)}_{\alpha,\beta} + u^{(0)}_{\beta,\alpha} - 2x_3u^{(0)}_{3,\alpha\beta} + \sum_{r=3}^{R}\left(\mathcal{L}^{(r)}_{\alpha\gamma}u^{(r)}_{\gamma,\beta} + \mathcal{L}^{(r)}_{\beta\gamma}u^{(r)}_{\gamma,\alpha}\right) + u^{(0)}_{3,\alpha}u^{(0)}_{3,\beta}\end{aligned} \tag{70}$$

$$\begin{aligned}2\tilde{e}_{\alpha3} &= \sum_{r=2}^{R}P^{(r)}_{\alpha\beta,3}u^{(r)}_\beta\\ &= \sum_{r=3}^{R}\mathcal{L}^{(r)}_{\alpha\beta,3}u^{(r)}_\beta\end{aligned} \tag{71}$$

Equations (68) do not formally differ from Eq. (8). The difference is that the quantities $\mathcal{L}^{(r)}_{\alpha\beta}(x_3)$ $(r \geq 2)$ are continuous functions in x_3 but with a piecewise-continuous first derivative, i.e., with a continuous derivative on the segments of a normal to the basic surface enclosed within the layers, i.e., in the interior of each layer, but with jumps at the interfaces between layers.

With the use of the stress–strain relations (2) and of the expression for the transverse shear strain components (71), one obtains

$${}^{(k)}\hat{U}_\alpha = {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda3\gamma3}\sum_{r=2}^{R}P^{(r)}_{\gamma\beta,3}\left({}^{(k)}Z^+\right)u^{(r)}_\beta \tag{72}$$

A. Equations of Motion and Boundary Conditions

Because of the constraint condition (64a), it is not possible to obtain the equations of motion and the boundary conditions directly from the virtual work statement in the form given by Eq. (30). Then we go back and elaborate on it further by taking into account the constraint condition (64a). The result is

$$
\begin{aligned}
&-\int_\Omega \Bigg(N_{\alpha\gamma,\alpha} - \hat{m}^{(0,0)}_{\beta\gamma}\ddot{u}^{(0)}_\beta + \hat{m}^{(1,0)}_{\beta\gamma}\ddot{u}^{(0)}_{3,\beta} - \sum_{r=2}^{R}\hat{m}^{(r,0)}_{\beta\gamma}\ddot{u}^{(r)}_\beta \\
&- \hat{m}^{(0)}_{\gamma\beta}\ddot{u}^{(0)}_{3,\beta}\Bigg)\delta u^{(0)}_\gamma \,\mathrm{d}\Omega - \int_\Omega \Bigg[M^{(1)}_{\alpha\gamma,\alpha\gamma} + \left(N_{\alpha\beta}u^{(0)}_{3,\alpha}\right)_{,\beta} + \bar{p}^t_3 + \bar{p}^b_3 \\
&- \left(\hat{m}^{(0,1)}_{\beta\gamma} - \hat{m}^{(0)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{\beta,\gamma} + \left(\hat{m}^{(1,1)}_{\beta\gamma} - \hat{m}^{(1)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta\gamma} \\
&- \sum_{r=2}^{R}\left(\hat{m}^{(r,1)}_{\beta\gamma} - \hat{m}^{(r)}_{\beta\gamma}\right)\ddot{u}^{(r)}_{\beta,\gamma} - \left(\hat{m}^{(1)}_{\gamma\beta} - \hat{m}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta\gamma} - m^{(0)}\ddot{u}^{(0)}_3 \Bigg] \\
&\times \delta u^{(0)}_3 \,\mathrm{d}\Omega - \sum_{s=2}^{R}\int_\Omega \Bigg(\hat{S}^{(s)}_{\alpha\gamma,\alpha} - T^{(s)}_\gamma - \hat{m}^{(0,s)}_{\beta\gamma}\ddot{u}^{(0)}_\beta + \hat{m}^{(1,s)}_{\beta\gamma}\ddot{u}^{(0)}_{3,\beta} \\
&- \sum_{r=2}^{R}\hat{m}^{(r,s)}_{\beta\gamma}\ddot{u}^{(r)}_\beta - \hat{m}^{(s)}_{\gamma\beta}\ddot{u}^{(0)}_{3,\beta}\Bigg)\delta u^{(s)}_\gamma \,\mathrm{d}\Omega \\
&- \int_{\Gamma_p}\Bigg[\sum_{s=0}^{R}\bar{\hat{S}}^{(s)}_{\gamma n}\delta u^{(s)}_\gamma + (\bar{V}_3 - \bar{\hat{M}}_{nt,t})\delta u^{(0)}_3 + \bar{\hat{M}}_{nn}\delta u^{(0)}_{3,n}\Bigg]\mathrm{d}\Gamma \\
&- \left[\bar{\hat{M}}_{nt}\delta u^{(0)}_3\right]_{|\Gamma_p} + \int_\Gamma\Bigg[\sum_{r=0}^{R}\hat{S}^{(r)}_{\gamma n}\delta u^{(r)}_\gamma + \left(N_{\alpha n}u^{(0)}_{3,\alpha} - \hat{V}_3 - \hat{M}_{nt,t}\right) \\
&\times \delta u^{(0)}_3 + \hat{M}_{nn}\delta u^{(0)}_{3,n}\Bigg]\mathrm{d}\Gamma + \left[\hat{M}_{nt}\delta u^{(0)}_3\right]_{|\Gamma} \\
&+ \int_\Gamma n_\gamma \Bigg[\hat{S}^{(1)}_{\alpha\gamma,\alpha} - \left(\hat{m}^{(0,1)}_{\beta\gamma} - \hat{m}^{(0)}_{\beta\gamma}\right)\ddot{u}^{(0)}_\beta + \left(\hat{m}^{(1,1)}_{\beta\gamma} - \hat{m}^{(1)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta} \\
&- \sum_{r=2}^{R}\left(\hat{m}^{(r,1)}_{\beta\gamma} - \hat{m}^{(r)}_{\beta\gamma}\right)\ddot{u}^{(r)}_\beta - \left(\hat{m}^{(1)}_{\gamma\beta} - \hat{m}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta}\Bigg]\delta u^{(0)}_3 \,\mathrm{d}\Gamma = 0
\end{aligned}
\tag{73}
$$

where use has been made of the relations (60) and (62). Thus, we obtain the following equations of motion.

For $\delta u^{(0)}_\gamma$,

$$
N_{\alpha\gamma,\alpha} = \hat{m}^{(0,0)}_{\beta\gamma}\ddot{u}^{(0)}_\beta - \left(\hat{m}^{(1,0)}_{\beta\gamma} - \hat{m}^{(0)}_{\gamma\beta}\right)\ddot{u}^{(0)}_{3,\beta} + \sum_{r=2}^{R}\hat{m}^{(r,0)}_{\beta\gamma}\ddot{u}^{(r)}_\beta \tag{74}
$$

For $\delta u^{(0)}_3$,

$$
\begin{aligned}
M^{(1)}_{\alpha\gamma,\alpha\gamma} + \left(N_{\alpha\beta}u^{(0)}_{3,\alpha}\right)_{,\beta} &= -\bar{p}^t_3 - \bar{p}^b_3 + m^{(0)}\ddot{u}^{(0)}_3 \\
&+ \left(\hat{m}^{(0,1)}_{\beta\gamma} - \hat{m}^{(0)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{\beta,\gamma} - \left(\hat{m}^{(1,1)}_{\beta\gamma} - \hat{m}^{(1)}_{\beta\gamma} - \hat{m}^{(1)}_{\gamma\beta} + \hat{m}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta\gamma} \\
&+ \sum_{r=2}^{R}\left(\hat{m}^{(r,1)}_{\beta\gamma} - \hat{m}^{(r)}_{\beta\gamma}\right)\ddot{u}^{(r)}_{\beta,\gamma}
\end{aligned}
\tag{75}
$$

For $\delta u^{(s)}_\gamma$,

$$
\hat{S}^{(s)}_{\alpha\gamma,\alpha} - T^{(s)}_\gamma = \hat{m}^{(0,s)}_{\beta\gamma}\ddot{u}^{(0)}_\beta - \left(\hat{m}^{(1,s)}_{\beta\gamma} - \hat{m}^{(s)}_{\gamma\beta}\right)\ddot{u}^{(0)}_{3,\beta} + \sum_{r=2}^{R}\hat{m}^{(r,s)}_{\beta\gamma}\ddot{u}^{(r)}_\beta\ddot{u}^{(0)}_{3,\beta} \tag{76}
$$

By taking into account that on Γ_u the virtual variations of the generalized displacements are zero, the contribution from the boundary integrals can be written as follows:

$$
\begin{aligned}
&\int_{\Gamma_p}(N_{\gamma n} - \bar{N}_{\gamma n})\delta u^{(0)}_\gamma \,\mathrm{d}\Gamma - \int_{\Gamma_p}\left(\hat{S}^{(1)}_{\gamma n} - \bar{\hat{S}}^{(1)}_{\gamma n}\right)\delta u^{(0)}_{3,\gamma}\,\mathrm{d}\Gamma \\
&+ \int_{\Gamma_p}\sum_{s=0}^{R}\left(\hat{S}^{(s)}_{\gamma n} - \bar{\hat{S}}^{(s)}_{\gamma n}\right)\delta u^{(s)}_\gamma \,\mathrm{d}\Gamma + \int_{\Gamma_p}(\hat{M}_{nn} - \bar{\hat{M}}_{nn})\delta u^{(0)}_{3,n}\,\mathrm{d}\Gamma \\
&+ \int_{\Gamma_p}\Bigg\{ N_{\alpha n}u^{(0)}_{3,\alpha} - \hat{V}_3 - \hat{M}_{nt,t} - \bar{V}_3 + \bar{\hat{M}}_{nt,t} \\
&+ n_\gamma\Bigg[M^{(1)}_{\alpha\gamma,\alpha} + \hat{M}_{\alpha\gamma,\alpha} - \left(\hat{m}^{(0,1)}_{\beta\gamma} - \hat{m}^{(0)}_{\beta\gamma}\right)\ddot{u}^{(0)}_\beta \\
&+ \left(\hat{m}^{(1,1)}_{\beta\gamma} - \hat{m}^{(1)}_{\beta\gamma} - \hat{m}^{(1)}_{\gamma\beta} + \hat{m}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta} \\
&- \sum_{r=2}^{R}\left(\hat{m}^{(r,1)}_{\beta\gamma} - \hat{m}^{(r)}_{\beta\gamma}\right)\ddot{u}^{(r)}_\beta \Bigg]\Bigg\}\delta u^{(0)}_3 \,\mathrm{d}\Gamma = 0
\end{aligned}
\tag{77}
$$

B. Third-Order Plate Model

This plate model has been proposed recently by Cheng et al.[15] We can obtain it by letting $R = 3$ in the equations derived in the preceding subsection. Specifically, the following results hold.

1. Kinematics

Substituting Eq. (66) with $p = 2$ into Eqs. (68), (70), and (71) yields

$$
\begin{aligned}
\tilde{u}_\alpha(x_j) &= u^{(0)}_\alpha - x_3u^{(0)}_{3,\alpha} + x_3^2\left(x_3\delta_{\alpha\beta} + \mathcal{C}^{(23)}_{\alpha\beta}\right)u^{(3)}_\beta \\
&+ u^{(3)}_\beta\mathcal{C}^{(23)}_{\alpha\beta}\sum_{r=2}^{3}\sum_{k=1}^{N-1}\Bigg[{}^{(k)}a^{(r)}_{\alpha\beta}\left(x_3 - {}^{(k)}Z^+\right) + {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda3\gamma3} \\
&\times\left(\delta_{\gamma\beta}r\left[{}^{(k)}Z^+\right]^{r-1} + \sum_{p=1}^{k-1}{}^{(p)}a^{(r)}_{\gamma\beta}H_p\right)\Bigg]H_k
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
2\tilde{e}_{\alpha\beta} &= u^{(0)}_{\alpha,\beta} + u^{(0)}_{\beta,\alpha} - 2x_3u^{(0)}_{3,\alpha\beta} + x_3^2\left(x_3\delta_{\alpha\gamma} + \mathcal{C}^{(23)}_{\alpha\gamma}\right)u^{(3)}_{\gamma,\beta} \\
&+ x_3^2\left(x_3\delta_{\beta\gamma} + \mathcal{C}^{(23)}_{\beta\gamma}\right)u^{(3)}_{\gamma,\alpha} + u^{(3)}_{\tau,\beta}\mathcal{C}^{(23)}_{\alpha\tau}\sum_{r=2}^{3}\sum_{k=1}^{N-1} \\
&\times\Bigg[{}^{(k)}a^{(2)}_{\alpha\tau}\left(x_3 - {}^{(k)}Z^+\right) + {}^{(k)}T_{\alpha\lambda}\,{}^{(k)}Q_{\lambda3\gamma3} \\
&\times\left(\delta_{\gamma\tau}r\left[{}^{(k)}Z^+\right]^{r-1} + \sum_{p=1}^{k-1}{}^{(p)}a^{(r)}_{\gamma\tau}H_p\right)\Bigg]H_k \\
&+ u^{(3)}_{\tau,\alpha}\mathcal{C}^{(23)}_{\beta\tau}\sum_{r=2}^{3}\sum_{k=1}^{N-1}\Bigg[{}^{(k)}a^{(2)}_{\beta\tau}\left(x_3 - {}^{(k)}Z^+\right) + {}^{(k)}T_{\beta\lambda}\,{}^{(k)}Q_{\lambda3\gamma3} \\
&\times\left(\delta_{\gamma\tau}r\left[{}^{(k)}Z^+\right]^{r-1}\sum_{p=1}^{k-1}{}^{(p)}a^{(2)}_{\gamma\tau}H_p\right)\Bigg]H_k + u^{(0)}_{3,\alpha}u^{(0)}_{3,\beta}
\end{aligned}
\tag{79}
$$

$$
2\tilde{e}_{\alpha3} = x_3\left(2\mathcal{C}^{(23)}_{\alpha\beta} + 3x_3\delta_{\alpha\beta}\right)u^{(3)}_\beta + u^{(3)}_\beta\mathcal{C}^{(23)}_{\alpha\beta}\sum_{r=2}^{3}\sum_{k=1}^{N-1}{}^{(k)}a^{(r)}_{\alpha\beta}H_k \tag{80}
$$

2. Equations of Motion

We obtain the equations of motion for this plate model exploiting the principle of virtual work in the form given by Eq. (73) and making use of the constraint condition (64b). The result is as follows.

For $\delta u^{(0)}_\gamma$,

$$
N_{\alpha\gamma,\alpha} = \hat{m}^{(0,0)}_{\beta\gamma}\ddot{u}^{(0)}_\beta - \hat{m}^{(1,0)}_{\beta\gamma}\ddot{u}^{(0)}_{3,\beta} + \left(\mathcal{C}^{(23)}_{\beta\omega}\hat{m}^{(2,0)}_{\beta\gamma} + \hat{m}^{(3,0)}_{\omega\gamma}\right)\ddot{u}^{(3)}_\omega \tag{81}
$$

For $\delta u^{(0)}_3$,

$$
\begin{aligned}
M^{(1)}_{\alpha\gamma,\alpha\gamma} + \left(N_{\alpha\beta}u^{(0)}_{3,\alpha}\right)_{,\beta} &= -\bar{p}^t_3 - \bar{p}^b_3 + \left(\hat{m}^{(0,1)}_{\beta\gamma} - \hat{m}^{(0)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{\beta,\gamma} \\
&- \left(\hat{m}^{(1,1)}_{\beta\gamma} - \hat{m}^{(1)}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta\gamma} + \left[\mathcal{C}^{(23)}_{\beta\omega}\left(\hat{m}^{(2,1)}_{\beta\gamma} - \hat{m}^{(2)}_{\beta\gamma}\right)\right. \\
&\left.+ \left(\hat{m}^{(3,1)}_{\omega\gamma} - \hat{m}^{(3)}_{\omega\gamma}\right)\right]\ddot{u}^{(3)}_{\omega,\gamma} + \left(\hat{m}^{(1)}_{\gamma\beta} - \hat{m}_{\beta\gamma}\right)\ddot{u}^{(0)}_{3,\beta\gamma} + m^{(0)}\ddot{u}^{(0)}_3
\end{aligned}
\tag{82}
$$

For $\delta u^{(3)}_\omega$,

$$
\begin{aligned}
\mathcal{C}^{(23)}_{\gamma\omega}\hat{S}^{(2)}_{\alpha\gamma,\alpha} + \hat{S}^{(3)}_{\alpha\omega,\alpha} - \mathcal{C}^{(23)}_{\gamma\omega}T^{(s)}_\gamma - T^{(3)}_\omega &= \left(\mathcal{C}^{(23)}_{\gamma\omega}\hat{m}^{(0,2)}_{\beta\gamma} + \hat{m}^{(0,3)}_{\beta\omega}\right)\ddot{u}^{(0)}_\beta \\
&- \left(\mathcal{C}^{(23)}_{\gamma\omega}\hat{m}^{(1,2)}_{\beta\gamma} + \hat{m}^{(1,3)}_{\beta\omega}\right)\ddot{u}^{(0)}_{3,\beta} + \sum_{r=2}^{3}\left(\mathcal{C}^{(23)}_{\gamma\omega}\hat{m}^{(r,2)}_{\beta\gamma} + \hat{m}^{(r,3)}_{\beta\gamma}\right)\ddot{u}^{(r)}_\beta
\end{aligned}
\tag{83}
$$

The corresponding linearized (small deflection theory) equations were derived recently by Cheng et al.[15]

C. First-Order Plate Models

If we neglect in Eq. (57) the contribution associated with $r \geq 2$, we obtain the displacement field employed by Schmidt and Librescu[14]:

$$\tilde{u}_\alpha(x_j) = u_\alpha^{(0)} + x_3 u_\alpha^{(1)} + \left(u_\beta^{(1)} + u_{3,\beta}^{(0)}\right)\hat{h}_{\alpha\beta} \tag{84}$$

The related equations of motion in terms of force and moment stress resultants follows in a straightforward manner from Eqs. (37) and (38), by taking into account the earlier relations. We obtain, in general, the following.

For $\delta u_\gamma^{(0)}$,

$$N_{\alpha\gamma,\alpha} = \hat{m}_{\beta\gamma}^{(0,0)}\ddot{u}_\beta^{(0)} + \hat{m}_{\beta\gamma}^{(1,0)}\ddot{u}_\beta^{(1)} + \sum_{r=2}^{R}\hat{m}_{\beta\gamma}^{(r,0)}\ddot{u}_\beta^{(r)} + \hat{m}_{\gamma\beta}^{(0)}\ddot{u}_{3,\beta}^{(0)} \tag{85}$$

For $\delta u_\gamma^{(1)}$,

$$\hat{S}_{\alpha\gamma,\alpha}^{(1)} - T_\gamma^{(1)} = \hat{m}_{\beta\gamma}^{(0,1)}\ddot{u}_\beta^{(0)} + \hat{m}_{\beta\gamma}^{(1,1)}\ddot{u}_\beta^{(1)} + \sum_{r=2}^{R}\hat{m}_{\beta\gamma}^{(r,1)}\ddot{u}_\beta^{(r)} + \hat{m}_{\gamma\beta}^{(1)}\ddot{u}_{3,\beta}^{(0)} \tag{86}$$

For $\delta u_\gamma^{(s)}$,

$$\hat{S}_{\alpha\gamma,\alpha}^{(s)} - T_\gamma^{(s)} = \hat{m}_{\beta\gamma}^{(0,s)}\ddot{u}_\beta^{(0)} + \hat{m}_{\beta\gamma}^{(1,s)}\ddot{u}_\beta^{(1)} + \sum_{r=2}^{R}\hat{m}_{\beta\gamma}^{(r,s)}\ddot{u}_\beta^{(r)} + \hat{m}_{\gamma\beta}^{(s)}\ddot{u}_{3,\beta}^{(0)} \qquad s \geq 2 \tag{87}$$

For $\delta u_3^{(0)}$,

$$\hat{M}_{\alpha\gamma,\alpha\gamma} - (Q_\alpha + T_\alpha)_{,\alpha} - \left(N_{\alpha\beta}u_{3,\alpha}^{(0)}\right)_{,\beta} = \bar{p}_3^t + \bar{p}_3^b + \hat{m}_{\gamma\beta}^{(0)}\ddot{u}_{\beta,\gamma}^{(0)} + \hat{m}_{\gamma\beta}^{(1)}\ddot{u}_{\beta,\gamma}^{(1)} + \sum_{r=2}^{R}\hat{m}_{\gamma\beta}^{(r)}\ddot{u}_{\beta,\gamma}^{(r)} + \hat{m}_{\beta\gamma}\ddot{u}_{3,\beta\gamma}^{(0)} - m^0\ddot{u}_3^{(0)} \tag{88}$$

In particular, for the case at hand, we have the following.

For $\delta u_\gamma^{(0)}$,

$$N_{\alpha\gamma,\alpha} = \hat{m}_{\beta\gamma}^{(0,0)}\ddot{u}_\beta^{(0)} + \hat{m}_{\beta\gamma}^{(1,0)}\ddot{u}_\beta^{(1)} + \hat{m}_{\gamma\beta}^{(0)}\ddot{u}_{3,\beta}^{(0)} \tag{89}$$

For $\delta u_\gamma^{(1)}$,

$$\hat{S}_{\alpha\gamma,\alpha}^{(1)} - T_\gamma^{(1)} = \hat{m}_{\beta\gamma}^{(0,1)}\ddot{u}_\beta^{(0)} + \hat{m}_{\beta\gamma}^{(1,1)}\ddot{u}_\beta^{(1)} + \hat{m}_{\gamma\beta}^{(1)}\ddot{u}_{3,\beta}^{(0)} \tag{90}$$

For $\delta u_3^{(0)}$,

$$\hat{M}_{\alpha\gamma,\alpha\gamma} - (Q_\alpha + T_\alpha)_{,\alpha} - \left(N_{\alpha\beta}u_{3,\alpha}^{(0)}\right)_{,\beta} = \bar{p}_3^t + \bar{p}_3^b + \hat{m}_{\gamma\beta}^{(0)}\ddot{u}_{\beta,\gamma}^{(0)} + \hat{m}_{\gamma\beta}^{(1)}\ddot{u}_{\beta,\gamma}^{(1)} + \hat{m}_{\beta\gamma}\ddot{u}_{3,\beta\gamma}^{(0)} - m^{(0)}\ddot{u}_3^{(0)} \tag{91}$$

D. Plate Models with Perfectly Bonded Interfaces

If in the displacement field (7) and in the subsequent derivation we neglect the effect due to interlayer slips, $\hat{U}_\alpha(x_j)$, we obtain the governing equations for multilayered plates with general layup and perfectly bonded interfaces. These plate models were formulated and discussed by Di Sciuva[16] using small deflection theory.

VI. Concluding Remarks

We formulate a general theory for the analysis of the geometrically nonlinear elastodynamic behavior of multilayered plates featuring interlayer slips. The theory is quite general in that no a priori assumption is made on the order of the thickness expansion of the in-plane displacements. The pertinent equations of motion and consistent boundary conditions are derived by means of the dynamic version of the principle of virtual work.

The generalization achieved by the proposed approach is assessed by deriving, as particular cases, the first-order theory proposed by Schmidt and Librescu[14] and the third-order theory proposed by Cheng et al.[15] for laminated plate featuring interlayer slips.

It is hoped that the present paper will contribute to a better understanding of the behavior of laminated plates with interfacial defects.

Work is in progress to obtain numerical results on the large deflection and postbuckling behavior of such plates by using models of different order. Another line of research is the development of finite elements based on the plate theory presented, to account for variation of the slip phenomenon in the plane of the plate.

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